数理解析研究所講究録
第 898 巻 1995 年 74-93

Linearised Stability for Abstract Quasilinear Evolution Equations of Paravolic Type II, Time Non Homogeneous Case

Kiyoko FURUYA and Atsushi YAGI

古谷 希世子 (お茶の水女子大学·理)
八木 厚志 (姫路工業大学·理)

1 Introduction.

We shall study the linearized stability of an abstract quasilinear evolution equation

$$(Q)\quad \begin{cases} du/dt + A(t,u)u = f(t,u), & 0 < t < \infty, \\ u(0) = u_0, & \end{cases}$$

of parabolic type in a Banach space X. Here, $-A(t,u)$ are the generators of analytic semigroups on X which are defined for $(t,u) \in I \times U$, where $I = [0,\infty)$ and $U = \{u \in Z; \|u\|_Z < R\}$ $(0 < R < \infty)$, Z being another Banach space continuously embedded in X with $\|\cdot\|_X \leq \|\cdot\|_Z$. The domains $\mathcal{D}(A(t,u))$ (which may not be dense in X) are allowed to vary with (t,u). $f(t,u)$ is an X-valued function defined for $(t,u) \in I \times U$ such that $f(t,0) = 0$ for $t \in I$. u_0 is an initial value in U. And $u = u(t)(0 \leq t < \infty)$ is an unknown function.

In the previous paper [8] we have already studied the stability of (Q) in the case that the equation in (Q) is time homogeneous, that is, in the case that $A(t,u) = A(u)$ and $f(t,u) = f(u)$. Under suitable conditions on $A(u)$ and $f(u)$ which guarantee the existence

and uniqueness of local solution, we have in fact proved that the following two Conditions:
(Sp) $\rho(A(0)) \supset \{\lambda \in \mathbf{C}; Re\lambda \leq \delta\}$ with some $\delta > 0$;
(f.ii)

$$\frac{\|f(u)\|_X}{\|u\|_Z} \to 0 \quad \text{as} \quad u \to 0 \quad \text{in} \quad Z;$$

yield the asymptotic stability of the zero stationary solution to (Q). By this we obtained a new result on linearized stability of the problem (Q) which is, the authors believe (see [8,Introduction]), more favorable in application than those known before, Amann [1,2,3,4], Drangeid [5]. In this paper we shall proceed to handle the time non homogeneous case and shall establish an analogous result on linearized stability of the (non homogeneous) problem (Q).

Let us here announce the Conditions we shall assume in this paper. But, before that, we may introduce three more Banach spaces $Y_i, i = 1,2,3$, such that $Z \subset Y_1 \subset X$ and that $Z \subset Y_3 \subset Y_2 \subset X$.

Assumptions on $A(t,u)$:
(A.i) The resolvent sets $\rho(A(t,u))$ of $A(t,u), (t,u) \in I \times U$, contain a sector $\Sigma = \left\{\lambda \in \mathbf{C}; |arg(\lambda - \omega)| \geq \theta_0 \text{ or } \lambda = \omega\right\}$, where $-\infty < \omega < \infty$ and $0 < \theta_0 < \pi/2$, and there the resolvents satisfy:

$$\|(\lambda - A(t,u))^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|\lambda - \omega| + 1}, \quad \lambda \in \Sigma, \quad (t,u) \in I \times U,$$

with some constant M.
(A.ii) For some $0 < \mu_0, \nu_0, \nu_1 \leq 1$,

$$\|(\omega - A(t,u))(\lambda - A(t,u))^{-1}[(\omega - A(t,u))^{-1} - (\omega - A(s,v))^{-1}]\|_{\mathcal{L}(X)}$$
$$\leq N_1 \left\{ \frac{h(|t-s|)^{\mu_0}}{(|\lambda - \omega| + 1)^{\nu_0}} + \frac{\|u - v\|_{Y_1}}{(|\lambda - \omega| + 1)^{\nu_1}} \right\},$$

$$\lambda \in \Sigma, \quad (t,u),(s,v) \in I \times U,$$

where $h(\tau) = \tau/(\tau+1), \tau \geq 0$,with some constant N_1. In addition,

$$(\omega - A(t,u))^{-1} - (\omega - A(t,v))^{-1} = R_2(t;u,v) + R_3(t;u,v),$$

and for some $0 < \nu_2 \leq \nu_3 \leq 1$

$$\|(\omega - A(t,u))(\lambda - A(t,u))^{-1} R_i(t;u,v)\|_{\mathcal{L}(X)} \leq N_i \frac{\|u-v\|_{Y_i}}{(|\lambda - \omega| + 1)^{\nu_i}},$$

$$\lambda \in \Sigma, \qquad (t,u),(t,v) \in I \times U, \quad i = 2,3,$$

with some constants $N_i (i = 2,3)$.
(A.iii) $A(t,0)$ has a limit $A(\infty,0)$ in the sense that

$$\|(\omega - A(t,0))^{-1} - (\omega - A(\infty,0))^{-1}\|_{\mathcal{L}(X)} \to 0, \quad \text{as} \quad t \to \infty,$$

where $A(\infty,0)$ is a closed linear operator in X with $\omega \in \rho(A(\infty,0))$.
Assumptions on the spaces $Z \subset Y_i \subset X (i = 1,2,3)$:
(S.i) For some $0 < \gamma_i < 1, \|\cdot\|_{Y_i} \leq \|\cdot\|_X^{\gamma_i} \|\cdot\|_Z^{1-\gamma_i}, i = 1,2,3$, on Z.
(S.ii) There is some $0 < \alpha < 1$ such that the domains of the fractional powers $[A(t,u) - \omega]^\alpha, (t,u) \in I \times U$, are contained in Z with continuous embedding:
$\|\cdot\|_Z \leq D\|[A(t,u) - \omega]^\alpha \cdot\|_X$ with some constant D.
(S.iii) There are some $0 \leq \alpha_2 \leq \alpha_3 \leq \alpha$ such that the domains of the fractional powers $[A(t,u) - \omega]^{\alpha_i}, (t,u) \in I \times U$, are contained in $Y_i, i = 2,3$, with continuous embedding:
$\|\cdot\|_{Yi} \leq D_i \|[A(t,u) - \omega]^{\alpha_i} \cdot\|_X$ with some constants D_i.
(S.iv) The unit ball $\{u \in Z; \|u\|_Z \leq 1\}$ of Z is closed with rerated to $(X, \|\cdot\|_X)$.
Relations among the exponents:

$$\text{(Ex)} \quad \mu_0 \geq \gamma_1 \alpha; \quad \nu_0 \geq \nu_1; \quad \gamma_1 \alpha + \nu_i > 1 \quad \text{for} \quad \mathrm{i} = 1,2,3;$$

$$\alpha+\nu_i>\alpha_i+1 \quad \text{for} \quad i=2,3; \quad \text{and} \quad \alpha(1+\gamma_2)+\nu_2>\alpha_3+1.$$

Spectral (or resolvent) Condition on $A(\infty,0)$:
(Sp) The resoluvent set $\rho(A(\infty,0))$ contains a half plane $\Lambda=\{\lambda\in \mathbf{C}; Re\lambda\leq\delta\}$ with some $\delta>0$.

Assumptions on $f(t,u)$:
(f.i) $\|f(t,u)-f(s,v)\|_X\leq L\{(\|u\|_Z+\|v\|_Z)|t-s|^\sigma+\|u-v\|_{Y_3}\}$, $(t,u),(s,v)\in I\times U$,
with some $0<\sigma\leq 1$ and with a constant L .
(f.ii)

$$\frac{\|f(t,u)\|_X}{\|u\|_Z}\to 0 \quad \text{as} \quad t\to\infty \qquad \text{and} \quad u\to 0 \quad \text{in} \quad Z.$$

Assumptions on the initial value u_0:
(In) u_0 belongs to $\mathcal{D}([A(0,u_0)-\omega]^\alpha)(\subset Z)$.

We shall argue with dividing the problem (Q) into two parts:

$$(Q_0)\begin{cases} du/dt+A(t,u)u=f(t,u), & 0<t\leq T,\\ u(0)=u_0, \end{cases}$$

$$(Q_T)\begin{cases} du/dt+A(t,u)u=f(t,u), & T<t<\infty,\\ u(T)=u_T, \end{cases}$$

where $0<T<\infty$ is some point fixed sufficiently large. As for (Q_0) it is already known by [17] that (A.i)(A.ii),(S.i)–(S.iv),(Ex) and (f.i) provide, for each sufficiently small u_0, existence of a unique global solution on $[0,T]$. So that, the problem of stability is reduced to studying (Q_T). As for (Q_T), then, we shall observe that the techniques we devised in [8] for homogeneous case are still available in proving, under (A.iii),(Sp) and (f.ii),existence and decay of a global solution on $[T,\infty)$ for each sufficiently small u_T.

As will be commented in Section 6, our result on the abstract equation (Q) actually applies to quasilinear parabolic partial differential equations.

Notations. The notations used in this paper are generally the same as before, so we shall refer the reader to Notation in [8,Introduction]. For example, by C we shall denote a universal constant which may change in each occurrence and which is determined in a specific way by the quantities in (A.i)–(A.iii),(S.i)–(S.iv),(Ex),(Sp) (f.i) and (f.ii). If C depends on some parameter, say θ, however, it will be denoted by C_θ.

By η_0 we shall denote a particular exponent given by

$$\eta_0 = Max\{\frac{1-\nu_i}{\gamma_1}, i=1,2; \quad \frac{\alpha_3+1-\alpha-\nu_2}{\gamma_2}\}(<\alpha \quad \text{from} \quad (\text{Ex})).$$

2 Consequence of the Conditions (Sp) and (A.iii).

In this section we shall notice that, for sufficiently large t and for sufficiently small $u, A(t,u)$ also satisfy the same spectral condition (Sp).

Proposition 2.1 *The Conditions (A.iii) and (Sp) jointed with (A.i), (A.ii) and (S.i) imply that, if* $0 < T_0 < \infty$ *is sufficiently large and if* $0 < R_0 < R$ *is sufficiently small, then,for all* $(t,u) \in [T_0,\infty) \times \{u \in Z; \|u\|_Z < R_0\}$, $\rho(A(t,u))$ *contain the half plain* Λ.

Proof. See [9].

We next notice:

Proposition 2.2 *(A.iii) jointed with (A.i) and (A.ii) implies that*

$$\begin{aligned} &\|A(t,u)(\lambda - A(t,u))^{-1}[A(t,u)^{-1} - A(s,v)^{-1}]\|_{\mathcal{L}(X)} \\ &\quad \leq (M+1)\{N(t)+N(s)+N_1(\|u\|_Z+\|v\|_Z)\}, \\ &\quad\quad (t,u),(s,v) \in [T_0,\infty) \times U, \end{aligned}$$

where $N(\cdot)$ *is a non negative function on* $[T_0,\infty)$ *such that* $\lim_{t\to\infty} N(t) = 0$.

Proof. See [9].

3 Solution on a finite interval $[0, T]$.

In this section we shall consider the equation (Q) in a finite interval $[0, T]$

$$(Q_0)\begin{cases} du/dt + A(t,u)u = f(t,u), & 0 < t \leq T, \\ u(0) = u_0, \end{cases}$$

here $0 < T < \infty$ is an arbitrarily fixed point.

According to [17,Sec.3], we have:

Proposition 3.1 *Let $0 < T < \infty$ be arbitrarily fixed. For any function space:*

$$(F_\omega; r, k, \eta) \quad \begin{cases} \|[A(t,u(t)) - \omega]^\alpha u(t)\|_X \leq r, & 0 \leq t \leq T, \\ \|u(t) - u(s)\|_X \leq k|t-s|^\eta, & 0 \leq s, t \leq T, \end{cases}$$

where $r > 0, k > 0$ and $\eta_0 < \eta < \alpha$, there exists $\varepsilon > 0$ such that, for each initial value u_0 satisfying (In) with $\|[A(0,u_0) - \omega]^\alpha u_0\|_X \leq \varepsilon$, (Q_0) possesses a unique solution $u \in C^1((0,T];X)$ on $[0,T]$ lying in $(F_\omega; r, k, \eta)$ (ε may depend on T). In addition, u satisfies:

$$\|[A(t,u(t)) - \omega]^\alpha u(t)\|_X \leq C\|[A(0,u_0) - \omega]^\alpha u_0\|_X, \quad 0 \leq t \leq T.$$

Proof. Indeed, if $\omega < 0$, let us change the unknown function to $v = e^{\omega t}u$. Then (Q_0)amounts to

$$(Q_0')\begin{cases} dv/dt + [A(t, e^{-\omega t}v) - \omega]v = e^{\omega t}f(t, e^{-\omega t}v), & 0 < t \leq T, \\ v(0) = u_0. \end{cases}$$

And the new coefficient operators $A(t, e^{-\omega t}v) - \omega$ defined for $(t, v) \in [0,T] \times e^{\omega T}U$ satisfy the Conditions (A.i),(A.ii),(S.ii) and (S.iii) with $\omega = 0$. Then the Proposition is an immediate consequence of [17,Theorem 3.3].

4 Solutions on an interval $[T, \infty)$.

Let T_0 be the point determined in Section 2. In view of the preceding section, we shall now consider the equation (Q) in a half line $[T, \infty), T_0 \le T < \infty$, that is, the problem:

$$(Q_T) \begin{cases} du/dt + A(t,u)u = f(t,u), & T < t < \infty, \\ u(T) = u_T. \end{cases}$$

The initial value u_T is assumed to satisfy:
(In_T) $\qquad u_T \in \mathcal{D}([A(T, u_T)]^\alpha)$.

Let us begin with noticing:

Proposition 4.1 *For any function space:*

$$(F; r', k', \eta') \quad \begin{cases} \|[A(t, u(t))]^\alpha u(t)\|_X \le r', & T \le t < \infty, \\ \|u(t) - u(s)\|_X \le k'|t-s|^{\eta'}, & T \le s, t < \infty, \end{cases}$$

where $r', k' > 0$ and $\eta_0 < \eta' < \alpha$, there exists $0 < \varepsilon' \le r'$ such that, for each initial value u_T satisfying (In_T) with $\|[A(T, u_T)]^\alpha u_T\|_X \le \varepsilon'$, (Q_T) possesses a unique local solution $u \in C^1((T, T+S]; X)$ lying in $(F; r', k', \eta')$, the length $S > 0$ being uniform in u_T. In addition, ε' and S are independent of the initial point $T \ge T_0$.

Proof. In fact this result has been already shown in [17,Proposition 3.1].

Let us next verify a priori estimates of the local solution to (Q_T):

Proposition 4.2 *One can take an initial point $T \in [T_0, \infty)$ and a function space:*

$$(F; R'', K'', \eta'') \quad \begin{cases} \|[A(t, u(t))]^\alpha u(t)\|_X \le R'', & T \le t < \infty, \\ \|u(t) - u(s)\|_X \le K''|t-s|^{\eta''}, & T \le s, t < \infty, \end{cases}$$

where $R'', K'' > 0$ *and* $\eta_0 < \eta'' < \alpha$, *as the following statement holds. For any function subspace:*

$$(F; r'', k'', \eta'') \quad \begin{cases} \|[A(t,u(t))]^\alpha u(t)\|_X \le r'', & T \le t < \infty, \\ \|u(t) - u(s)\|_X \le k''|t-s|^{\eta''}, & T \le s,t < \infty, \end{cases}$$

where $0 < r'' \le R''$ *and* $0 < k'' \le K''$, *there exists* $0 < \varepsilon'' \le r''$ *such that, for every local solution* $u \in C^1((T, T+S(u)]; X)$ *to* (Q_T) *which lies in* $(F; R'', K'', \eta'')$ *on* $[T, T+S(u)]$, *if* u_T *satisfies:* $\|[A(T,u_T)]^\alpha u_T\|_X \le \varepsilon''$, *then actually* u *lies in* $(F; r'', k'', \eta'')$ *on* $[T, T+S(u)]$, ε'' *being independent of the length* $S(u)$.

Proof. See [9].

From these Propositions we obtain:

Theorem 4.1 *Under (A.i)–(A.iii),(S.i)–(S.iv),(Ex),(Sp),(f.i) and (f.ii), one can take an initial point* $T \in [T_0, \infty)$ *and a function space:*

$$(F; r, k, \eta) \quad \begin{cases} \|[A(t,u(t))]^\alpha u(t)\|_X \le r, & T \le t < \infty, \\ \|u(t) - u(s)\|_X \le k|t-s|^{\eta}, & T \le s,t < \infty, \end{cases}$$

where $r, k > 0$ *and* $\eta_0 < \eta < \alpha$, *as the following statement holds. There exists a number* $\varepsilon > 0$ *such that, for any initial value* u_T *satisfying* (In_T) *with* $\|[A(T,u_T)]^\alpha u_T\|_X \le \varepsilon$, (Q_T) *possesses a unique global solution* $u \in C^1((T,\infty); X)$ *on* $[T, \infty)$ *lying in* $(F; r, k, \eta)$ *and satisfying:*

$$\|[A(t,u(t))]^\alpha u(t)\|_X \le C e^{-(\delta/3)(t-T)} \|[A(T,u_T)]^\alpha u_T\|_X, \quad T \le t < \infty. \tag{1}$$

Proof. Let T, R'', K'' and η'' be the point and the numbers determined as in Proposition 4.2. Set the numbers r', k' and η' in Proposition 4.1 as $r' = R'', k' = K''/2$ and $\eta' = \eta''$; and, set $r'' = \varepsilon'$ and $k'' = K''/2$. Then the Theorem will be proved with

$r = r'', k = K''/2, \eta = \eta''$ and $\varepsilon = \varepsilon''$. In fact, let u_T satisfy: $\|[A(T, u_T)]^\alpha u_T\|_X \leq \varepsilon$. Then, since $\varepsilon \leq \varepsilon'$, Proposition 4.1 first yields existence of a solution u in $(F; r', k', \eta')$ on the interval $[T, T+S]$; but, since $r' = R'', k' \leq K''$ and $\varepsilon = \varepsilon''$, by virtue of Proposition 4.2, u actually lies in $(F; r''; k''; \eta'')$ and hence in $(F; r; k; \eta)$. Assume next that this solution can be extended as a solution $u \in C^1((T, T+S(u)]; X)$ on an interval, say $[T, T+S(u)]$ $(S(u) \geq S)$ in the space $(F; r; k; \eta)$. Take a point $T' = T + S(u) - S/2$(where S is the length determined in Proposition 4.1). Then, since $\|[A(T', u(T'))]^\alpha u(T')\|_X \leq r = \varepsilon'$, Proposition 4.1 yields again that there is an extension of u on the interval $[T, T'+S]$; it is easy to observe that the extended solution $\tilde{u}$ lies in $(F; R'', K'', \eta'')$ on the interval. By the same reason as above it then follows that $\tilde{u}$ lies in $(F; r, k, \eta)$. In this way we have obtained an extension of solution of $S/2$ length in the space $(F; r, k, \eta)$. As $S/2$ is uniform in each extension, we can continue this procedure and finally construct a global solution to(Q_T) in $(F; r, k, \eta)$. The estimate (6) was already verified by (5).

5 Asymptotic Stability of Zero Solution to (Q).

We are now in a position to prove the main result of this paper:

Theorem 5.1 *Assume (A.i)-(A.iii),(S.i)-(S.iv),(Ex),(Sp),(f.i) and (f.ii). Then, there exist a function space:*

$$(E; r, k, \eta) \quad \begin{cases} \|u(t)\|_Z \leq r, & 0 \leq t < \infty, \\ \|u(t) - u(s)\|_X \leq k|t-s|^\eta, & 0 \leq s, t < \infty, \end{cases}$$

where $r, k > 0$ and $\eta_0 < \eta < \alpha$, and a number $\varepsilon > 0$ such that, for any initial value u_0 satisfying (In) with $\|[A(0, u_0) - \omega]^\alpha u_0\|_X \leq \varepsilon$, *(Q) possesses a unique global solution* $u \in C^1((0,\infty);X)$ *on* $[0,\infty)$ *lying in* $(E;r,k,\eta)$. *Moreover, the solution decays as*

$$\|u(t)\|_Z \le Ce^{-(\delta/3)t}\|[A(0,u_0)-\omega]^\alpha u_0\|_X, \qquad 0 \le t < \infty, \tag{2}$$

C *being independent of* u_0.

Proof. This result follows immediately from Proposition 3.1 and Theorem 4.1. Only thing to be noticed here is that

$$\|[A(t,u)]^\alpha \cdot \|_X \le C\|[A(t,u)-\omega]^\alpha \cdot \|_X, \qquad (t,u) \in [T_0,\infty) \times U,$$

with some constant C; but, this is easily seen from [18,Chap. 2, Lemma 3.5].

Furthermore, the decay estimate (7) is actually improved as:

Theorem 5.2 *The global solution* u *constructed in Theorem 5.1 decays as, for any* $0 < \beta < \delta$,

$$\|A(t,u(t))u(t)\|_X \le C_\beta t^{\alpha-1}e^{-\beta t}\|[A(0,u_0)-\omega]^\alpha u_0\|_X, \qquad 0 < t < \infty, \tag{3}$$

C_β *being independent of* u_0.

Proof. The proof will be accomplished by three Steps.

Step 1. Set, as before, $A_u(t) = A(t,u(t)), 0 \le t < \infty$. As was verified by (4), for any $T \in [T_0,\infty)$,

$$\begin{aligned}&\|A_u(t)(\lambda - A_u(t))^{-1}\{A_u(t)^{-1} - A_u(s)^{-1}\}\|_{\mathcal{L}(X)} \\ &\quad \le C\{N(T) + \sup_{T\le t<\infty}\|u(t)\|_Z\}^\zeta \left\{\frac{|t-s|^{\gamma_1\eta}}{(|\lambda|+1)^{\nu_1}}\right\}^{1-\zeta}, \lambda \in \Sigma, T \le t < \infty,\end{aligned}$$

with any $0 < \zeta < 1$. While, in view of (7), $N(T)+\sup_{T\le t<\infty}\|u(t)\|_Z \to 0$ as $T \to \infty$. This then means, by virtue of Theorem A.2, that, for any $\beta < \delta' < \delta$, if T is sufficiently large, then the evolution operator $U_u(t,s)$ for $A_u(t)$ satisfies:

$$\|[A_u(t)]^\theta U_u(t,s)\|_{\mathcal{L}(X)} \le C_{\delta'}(t-s)^{-\theta}e^{-\delta'(t-s)}, T \le s \le t < \infty, \tag{4}$$

$$\|[A_u(t)]^\theta U_u(t,s)[A_u(s)]^{-\theta}\|_{\mathcal{L}(X)} \le C_{\delta'} e^{-\delta'(t-s)}, T \le s \le t < \infty, \quad (5)$$

for all $0 \le \theta \le 1$.

Step 2. We can then argue in the same way as in the second Step of proof of Proposition 4.2. And indeed we obtain that, if T is sufficiently large, then

$$\|[A_u(t)]^\alpha u(t)\|_X \le C_{\delta'} e^{-\beta(t-T)} \|[A_u(T)]^\alpha u(T)\|_X, T \le t < \infty, \quad (6)$$

$$\|u(t)-u(s)\|_X \le C_{\delta'}(t-s)^\eta e^{-\beta(s-T)} \|[A_u(T)]^\alpha u(T)\|_X, T \le s \le t < \infty. \quad (7)$$

Step 3. Let again $\beta < \delta' < \delta$. Since $0 < \beta < \delta$ was arbitrary in the Step 2, we can assume that (11) and (12) hold with δ' substituted for β. On the other hand, we verify from Theorem A.2 that

$$\begin{aligned} &\|A_u(t)U_u(t,s) - A_u(t)exp(-(t-s)A_u(t))\|_{\mathcal{L}(X)} \\ &\le C_{\delta'}(t-s)^{(1-\zeta)(\gamma_1\eta+\nu_1)-2} e^{-\delta'(t-s)}, \\ &T \le s \le t < \infty, \end{aligned} \quad (8)$$

if T is sufficiently large. In order to estimate $A_u(t)u(t)$, we shall now start with

$$u(t) = U_u(t,T)u(T) + \int_T^t U_u(t,\tau) f(\tau, u(\tau))d\tau, \qquad T \le t < \infty.$$

Operating $A_u(t)$, it is written in the form

$$A_u(t)u(t) = I + II + III + IV,$$

where

$$I = A_u(t)U_u(t,T)[A_u(T)]^{-\alpha}[A_u(T)]^\alpha u(T),$$

$$II = \int_T^t A_u(t)U_u(t,\tau)\{f(\tau,u(\tau)) - f(t,u(t))\}d\tau,$$

$$III = \int_T^t \{A_u(t)U_u(t,\tau) - A_u(t)exp(-(t-\tau)A_u(t))\}d\tau \cdot f(t,u(t)),$$

$$IV = \{1 - exp(-(t-T)A_u(t))\}f(t,u(t)).$$

After some calculation, we have verified the estimate (8) for large $t \geq T+1$. For $0 < t \leq T+1$, however, the proof is more immediate; indeed, by a similar argument we easily verify that

$$\|A_u(t)u(t)\|_X \leq Ct^{\alpha-1}\|[A(0,u_0)-\omega]^\alpha u_0\|_X, \qquad 0 < t \leq T+1.$$

6 Comment on Application.

As a model of application of our abstract result, we can consider the following quasilinear parabolic differential equation:

$$(D)\begin{cases} \partial u/\partial t + A(t,x,u;D)u = f(t,x,u,\nabla u) & \text{in} \quad (0,\infty)\times\Omega, \\ B(t,x,u;D)u = 0 \quad \text{on} \quad (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x) \quad \text{in} \quad \Omega, \end{cases}$$

in a bounded region $\Omega \subset \mathbf{R}^n$. Here,

$$A(t,x,u;D)v = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(t,x,u)\frac{\partial v}{\partial x_j} + c(t,x,u)v$$

are differential operators in Ω with real valued functions a_{ij} and c on $I \times \bar{\Omega} \times \mathbf{C}$, where $I = [0,\infty)$.

$$B(t,x,u;D)v = \sum_{i,j=1}^{n} a_{ij}(t,x,u)\nu_i(x)\frac{\partial v}{\partial x_j} + g(t,x,u)v$$

are boundary differential operators on $\partial\Omega$ with a real valued function g on $I \times \partial\Omega \times \mathbf{C}, \nu(x) = (\nu_1(x),\cdots,\nu_n(x))$ being the outer normal vector at $x \in \partial\Omega$. $f(t,x,u,q)$ is a complex valued function on $I \times \bar{\Omega} \times \mathbf{C} \times \mathbf{C}^n$. u_0 is an initial function in Ω . And $u = u(t,x), (t,x) \in (0,\infty) \times \bar{\Omega}$, is an unknown function.

We shall assume the following Conditions:
(Ω) Ω is a bounded region in $\mathbf{R}^n$ of C^2-class .

(a.i) $a_{ij} \in C(\bar{I}; C^{2l+1}(\bar{\Omega}\times(\mathbf{R}+i\mathbf{R})))\cap C^l(I; C^1(\bar{\Omega}\times(\mathbf{R}+i\mathbf{R}))), 1 \leq i,j \leq n$, with some $1/2 < l \leq 1$.
(a.ii) $a_{ij} = a_{ji}(1 \leq i,j \leq n)$, and there exists some $\varepsilon > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(t,x,u)q_i q_j \geq \varepsilon|q|^2, \quad q \in \mathbf{R}^n,$$

for each $(t,x,u) \in I \times \bar{\Omega} \times \mathbf{C}$.
(c) $c \in C(\bar{I}; C^{2l}(\bar{\Omega} \times (\mathbf{R}+i\mathbf{R}))) \cap C^l(I; C(\bar{\Omega} \times (\mathbf{R}+i\mathbf{R})))$ with some $1/2 < l \leq 1$.
(*f*.1) $f \in C(I \times \bar{\Omega} \times \mathbf{C} \times \mathbf{C}^n)$, and for some $0 < \sigma \leq 1$

$$|f(t,x,u,q)-f(s,x,v,r)| \leq L_G\{(|u|+|v|+|q|+|r|)|t-s|^\sigma+|u-v|+|q-r|\},$$

$$0 \leq s,t < \infty; \quad x \in \bar{\Omega} \quad ; (u,q),(v,r) \in G,$$

in each bounded subset $G \subset \mathbf{C} \times \mathbf{C}^n$.
(*f*.2)

$$\sup_{x\in\bar{\Omega}} \frac{|f(t,x,u,q)|}{|u|+|q|} \to 0 \qquad \text{as} \quad t \to \infty, \quad u \to 0 \quad \text{and} \quad q \to 0.$$

(g)$g \in C(\bar{I}; C^{2l}(\partial\Omega \times (\mathbf{R}+i\mathbf{R}))) \cap C^l(I; C(\partial\Omega \times (\mathbf{R}+i\mathbf{R})))$ with some $1/2 < l \leq 1$.

Notation. For $I = [0,\infty)$ and a Frechet space $\mathcal{X}$, $C(\bar{I};\mathcal{X})$ denotes the set of $\mathcal{X}$ valued continuous functions $\varphi(t)$ defined on I which have, as $t \to \infty$, limits $\varphi(\infty)$ in $\mathcal{X}$. $C^l(I;\mathcal{X})$ denotes the set of $\mathcal{X}$ valued functions $\varphi(t)$ on I which satisfy, for each seminorm μ, $\mu(\varphi(t)-\varphi(s)) \leq C_{\varphi,\mu}h(|t-s|)^l, 0 \leq s,t < \infty$, where $h(\tau) = \tau/(\tau+1), \tau \geq 0$.

Obviously (*f*.1) and (*f*.2) imply that $f(t,x,0,0) = 0$, so that $u \equiv 0$ is a stationary solution to (D). Our question is then examining when the zero solution is asymptotically stable.

Set

$$X = L_p(\Omega), \quad n < p < \infty, \quad \text{and} \quad Z = W_p^h(\Omega), \quad 1 + \frac{n}{p} < h < 2.$$

We shall then formulate (D) as an abstract equation:

$$(Q) \quad \begin{cases} du/dt + A(t,u)u = f(t,u), & 0 < t < \infty, \\ u(0) = u_0, \end{cases}$$

in X. Here

$$(\mathcal{A}) \quad \begin{cases} \mathcal{D}(A(t,u)) = \left\{ v \in W_p^2(\Omega); B(t,x,u;D)v = 0 \quad \text{on} \quad \partial\Omega \right\} \\ A(t,u)v = A(t,x,u;D)v \end{cases}$$

for $(t,u) \in \bar{I} \times U$, where $U = \{u \in Z; \|u\|_Z < R\}$ with some $0 < R < \infty$. $f(t,u) = f(t,x,u,\nabla u)$ for $(t,u) \in I \times U$. And, $u_0 \in W_p^a(\Omega)$ with $h < a < 2$. In addition, we set:

$$Y_1 = Y_2 = W_p^{h-1}(\Omega), \quad Y_3 = W_p^1(\Omega).$$

We shall next verify, in order to apply the Theorems 5.1 and 5.2, that the Conditions (A.i)–(A.iii),(S.i)–(S.iv),(Ex),(f.i)(f.ii) and (In) (except (Sp)) are all fulfilled. But essentially such verification has been already done through our previous papers [16, Sec.5], [17,Sec.4] and [8,Sec.6]. Let us recall that so that very briefly.

When u vary in U, u are contained in some compact subset of $C^1(\bar{\Omega})$ because of the embedding: $W_p^h(\Omega) \subset C^1(\bar{\Omega})$. So that, from (a.1), $a_{ij}(t,x,u)$ are also contained in some compact subset of $C^1(\bar{\Omega})$ when (t,u) vary in $I \times U$; similarly, from (g), $g(t,x,u)$ are contained in some compact subset of $C^1(\partial\Omega)$. Therefore, (A.i) is verified from the strong ellipticity (a.2). As was verified in [16,Sec.5] and in [17,Sec.4], (A.ii) holds with $\mu_0 = l$, with any $\nu_0 = \nu_1 < 1/2$, with any $\nu_2 < (p+1)/2p$ and with $\nu_3 = 1$ respectively. In the same way, (A.iii) is verified from the condition that the coefficients of $A(t,x,u;D)$ and $B(t,x,u;D)$ are continuous at $t=\infty$.

(S.i) holds with $\gamma_1=\gamma_2=1/h$ and with any $\gamma_3<(h-1)/h$. As was done in [15,Appendix], we can estimate the domains of the fractional powers $[A(t,u)-\omega]^\theta$, $0\leq\theta\leq 1$, by using the Sobolev spaces $W_p^{2\theta}(\Omega)$; according to this, (S.ii) and (S.iii) are the case provided $h/2<\alpha,(h-1)/2<\alpha_2$ and $1/2<\alpha_3$ respectively. (S.iv) is seen, on the other hand, from the sequentially weak compactness of the closed unit ball in the reflexive Banach space Z.

(Ex) is now evident. Indeed, it suffices to take $\alpha>h/2,\alpha_3>1/2$ and $\alpha_2>(h-1)/2$ sufficiently small and $\gamma_3<(h-1)/h,\nu_0=\nu_1<1/2$ and $\nu_2<(p+1)/2p$ sufficiently large respectively.

(f.i) is an immediate consequence of (f.1), since, when u vary in $U,u(x)$ and $|\nabla u(x)|$ are bounded on $\bar{\Omega}$. Similarly, (f.ii) follows from (f.2), since $\|u\|_Z\to 0$ implies that $u(x)$ and $|\nabla u(x)|\to 0$ on $\bar{\Omega}$.

Finally, in order to verify (In), we shall assume further that (u_0) The initial function u_0 belongs to $W_p^a(\Omega)$ with $a>2\alpha$ and satisfies the compatibility condition:

$$\sum_{i,j=1}^{n} a_{ij}(0,x,u_0)\nu_i(x)\frac{\partial u_0}{\partial x_j}+g(0,x,u_0)u_0=0 \quad \text{on} \quad \partial\Omega.$$

Then, (In) follows from (u_0) by virtue of [15,Theorem A.2].

As a conclusion we can state now that, under (Ω),(a.1),(a.2),(c), (g),(f.1) and (f.2), the stability problem of the zero solution is reduced to verifying the spectral Condition (Sp) of the operator $A(\infty,0)$ defined by ($\mathcal{A}$). That is, if $A(\infty,0)$ satisfies (Sp), then (D) possesses, for each sufficiently small initial function u_0 satisfying (u_0), a unique global solution which decays exponentially as $t\to\infty$.

It is similarly possible to examine the stability for another stationary solution $\bar{u}(\neq 0)$ to (D). Changing the unknown functions from u to $v = u - \bar{u}$, we shall rewrite (D) round $\bar{u}$. Then our abstract result will become applicable; for detailed procedure, however, we shall refer the reader to [8,Sec.6].

Appendix;Related Linear Equations.

In this section we shall consider an abstract linear evolution equation

$$(L) \qquad \begin{cases} du/dt + A(t)u = f(t), & 0 < t \le T, \\ u(0) = u_0, \end{cases}$$

in X . Here, $-A(t), 0 \le t \le T$, are the generators of analytic semigroups on X, $f : [0,T] \to X$ is a Hölder continuous function and $u_0 \in \overline{\mathcal{D}(A(0))}$ is an initial value.

It is already known that an evolution operator $U(t,s), 0 \le s \le t \le T$, for (L) can be constructed under the following three Conditions:

(L.A.i) The resolvent sets $\rho(A(t))(0 \le t \le T)$ of $A(t)$ contain $\Sigma = \{\lambda \in \mathbf{C}; |arg\lambda| \ge \theta_0\}$,

where $0 < \theta_0 < \pi/2$, and there the resolvents satisfy:

$$\|(\lambda - A(t))^{-1}\|_{\mathcal{L}(X)} \le \frac{M}{(|\lambda|+1)}, \quad \lambda \in \Sigma, \quad 0 \le t \le T,$$

with some constant M.

(L.A.ii) For some $0 < \mu, \nu \le 1$,

$$\|A(t)(\lambda - A(t))^{-1}[A(t)^{-1} - A(s)^{-1}]\|_{\mathcal{L}(X)} \le N\frac{|t-s|^{\mu}}{(|\lambda|+1)^{\nu}}, \qquad \lambda \in \Sigma, \quad 0 \le s,t \le T,$$

with some constant N.

(L.Ex) The exponents satisfy a relation: $\mu + \nu > 1$.

Theorem A.1 *Under (L.A.i),(L.A.ii) and (L.Ex), there exists bounded linear operators* $U(t,s), 0 \leq s \leq t \leq T$, *which provides a unique solution* $u \in C^1((0,T];X) \cap C([0,T];X)$ *to* (L) *by the formula*

$$u(t) = U(t,0)u_0 + \int_0^t U(t,\tau)f(\tau)d\tau, \qquad 0 \leq t \leq T,$$

For the proof, see [13,14].

Assume, in addition to the above three, the spectral Condition for each $A(t)$:
(L.Sp) $\rho(A(t)) \supset \Lambda = \{\lambda \in \mathbf{C}; Re\lambda \leq \delta\}, \delta > 0$, for $0 \leq t \leq T$; and $\|(\lambda - A(t))^{-1}\|_{\mathcal{L}(X)} \leq M$ for $\lambda \in \Lambda, 0 \leq t \leq T$.
Then the various decay estimates to $U(t,s)$ are established:

Theorem A.2 *Let (L.A.i),(L.A.ii),(L.Sp) and (L.Ex) be satisfied, and fix a* ρ *such that* $1-\mu < \rho < \nu$. *Then, for any* $0 < \delta' < \delta$, *the estimates:*

$$\|A(t)^\theta U(t,s)\|_{\mathcal{L}(X)} \leq C_{\delta'}(t-s)^{-\theta} exp\{(C_{\delta'}N^{1/(\rho+\mu-1)} - \delta')(t-s)\},$$

$$\|A(t)^\theta U(t,s)A(s)^{-\theta}\|_{\mathcal{L}(X)} \leq C_{\delta'} exp\{(C_{\delta'}N^{1/(\rho+\mu-1)} - \delta')(t-s)\},$$

$$\|\{U(t,s)-1\}A(s)^{-\theta}\|_{\mathcal{L}(X)} \leq C_{\delta'}(t-s)^\theta[exp\{(C_{\delta'}N^{1/(\rho+\mu-1)} - \delta')(t-s)\}+1],$$

$$\begin{aligned}&\|A(t)U(t,s) - A(t)exp(-(t-s)A(t))\|_{\mathcal{L}(X)} \\ &\qquad \leq C_{\delta'}(t-s)^{\mu+\nu-2} exp\{(C_{\delta'}N^{1/(\rho+\mu-1)} - \delta')(t-s)\},\end{aligned}$$

hold for $0 \leq \theta \leq 1$ *and* $0 \leq s < t \leq T$ *with some* $C_{\delta'}$ *which is independent of* N *and* T.

For the proof, see [8,Th.2.3].

References

[1] Amann, H.,Parabolic evolution equations in interpolation and extrapolation spaces,J.Funct.Anal.,**78**(1988),233 -270.

[2] Amann, H.,Dynamic theory of quasilinear parabolic equations-I. Abstract evolution equations,Nonlinear Anal.,Theory,Method & Appli,.**12**(1988),895-919.

[3] Amann, H.,Dynamic theory of quasilinear parabolic equations II. Reaction-diffusion systems,Differ.Inte.Equations,**3** (1990),13-75.

[4] Amann, H.,Dynamic theory of quasilinear parabolic systems-III. Global existence,Math.Z.,**202**(1989). 219-250.

[5] Drangeid, A. K.,The principle of linearized stability for quasilinear parabolic evolution equations,Nonlinear Anal.,Theory, Method & Appli.,**13**(1989),1091-1113.

[6] Furuya, K.,Analyticity of solutions of quasilinear evolution equations,Osaka J.Math.,**18**(1981),669-698.

[7] Furuya, K.,Analyticity of solutions of quasilinear evolution equations II,Osaka J.Math.,**20**(1983),217-236.

[8] Furuya, K. and Yagi, A.,Linearized stability for abstract quasilinear evolution equations of parabolic type,to appear in Funkcial.Ekvac.

[9] Furuya, K. and Yagi, A.,Linearized stability for abstract quasilinear evolution equations of parabolic type II,time non homogeneous case,preprint.

[10] Lunarudi. A.,Asymptotic exponential stability in quasilinear parabolic equations,Nonlinear Anal.,Theory,Method & Appli.,**9**(1985),563-586.

[11] Potier-Ferry, M.,The linearization principle for the stability of solutions of quasilinear parabolic equations I,Arch.rat. Mech.Anal.,**77**(1981),301-320.

[12] Sovolevskii, P. E.,Parabolic equations in Banach space with an unbounded variable operator, a fractional power of which has a constant domain of definition,Soviet Math.Docl.,**2**(1961),545-548.

[13] Yagi, A.,Fractional powers of operators and evolution equations of parabolic type ,Proc. Japan Acad.Ser.A,**64**(1988),227-230.

[14] Yagi, A.,Parabolic evolution equations in which the coefficients are the generator of infinitely differentiable semigroups, Funkcial.Ekvac.,**32**(1989),107-124.

[15] Yagi, A.,Parabolic evolution equations in which the coefficients are the generator of infinitely differentiable semigroups II,Funkcial.Ekvac.,**33**(1990),139-150.

[16] Yagi, A.,Abstract quasilinear evolution equations of parabolic type in Banach spaces,Bollentino U.M.I.,**5**-B(1991),341- 368.

[17] Yagi, A.,Abstract quasilinear evolution equations of parabolic type,II,preprint.

[18] Tanabe, H., Equations of Evolution,Tokyo,Iwanami,1975(in Japanese). English translation, London,Pitman,1979.

[19] Triebel, H., Interpolation theory, Function spaces, Differential operators;Amsterdam, North-Holland,1987.

Kiyoko FURUYA
Department of Mathematics
Ochanomizu University
2-1-1 Ōtsuka, bunkyou-ku, Tokyo
Japan
E-mail:furuya@math.ocha.ac.jp

Atsushi YAGI
Department of Mathematics
Himegi Institute of Technology
2167 Shosha, Himeji, Hyougo 671-22,
Japan